Unit 3: The Derivative

Name: ____________________

Understanding the Derivative as the Slope of a Tangent Line

Let's examine the slope of the tangent line for: $f(x) = x^2$ at the point (3,9).

Using the limit definition:

Using the slope formula:

The Derivative

The derivative of a function is the formula obtained by finding the:

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

The derivative yields the formula that can be used to find the
slope of the tangent line to the graph of the function at <u>any single point</u> on the graph.

<u>Examples:</u>

1) Find the derivative for $f(x) = 2x^2 + 3$ using the limit definition.
Use it to find the slope of the tangent line to the graph of $f(x)$ at x = 1.

2) Find the derivative for $g(x) = x^3 - 3x + 2$ using the limit definition.
Use it to find the slope of the tangent line to the graph of $g(x)$ at x = -2.

3) Find the derivative for $f(x) = \dfrac{1}{x-1}$ using the limit definition.
Use it to find the slope of the tangent line to the graph of $f(x)$ at x = -5.

4) Find the derivate of $f(x) = \sqrt{x}$

Examples:

1) Find the derivative for $f(x) = 2x^2 + 3$ using the limit definition.
   Use it to find the slope of the tangent line to the graph of $f(x)$ at x = 1.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{[2(x+h)^2 + 3] - [2x^2 + 3]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) + 3 - 2x^2 - 3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 3 - 2x^2 - 3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{4xh + 2h^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2h(2x+h)}{h}$$

$$f'(x) = \lim_{h\to 0} (4x + 2h) = 4x$$

$$f'(1) = 4(1) = 4$$

$\therefore$ m = 4 at (1,5)

2) Find the derivative for $g(x) = x^3 - 3x + 2$ using the limit definition.

Use it to find the slope of the tangent line to the graph of $g(x)$ at x = -2.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h) + 2] - [x^3 - 3x + 2]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 2 - x^3 + 3x - 2]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h}$$

$$f'(x) = \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3)$$

$$f'(x) = (3x^2 - 3)$$

$$f'(-2) = 9$$

at (-2,0) m = 9

$$y - y_1 = m(x - x_1)$$
$$y - 0 = 9(x - {}^{-}2)$$
$$y = 9(x + 2)$$
$$y = 9x + 18$$

3) Find the derivative for $f(x)=\dfrac{1}{x-1}$ using the limit definition.

Use it to find the slope of the tangent line to the graph of $f(x)$ at x = -5.

Hint: If the function is rational, you will probably need to find a common denominator to find the derivative.

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{x+h-1}-\dfrac{1}{x-1}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{(x+h-1)}\bullet\dfrac{(x-1)}{(x-1)}-\dfrac{1}{(x-1)}\bullet\dfrac{(x+h-1)}{(x+h-1)}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{1(x-1)}{(x+h-1)(x-1)}-\dfrac{1(x+h-1)}{(x-1)(x+h-1)}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{(x-1)-(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{x-1-x-h+1}{(x+h-1)(x-1)h}$$

$$f'(x)=\lim_{h\to 0}\frac{-h}{(x+h-1)(x-1)h}$$

$$f'(x)=\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}$$

$$f'(x)=\frac{-1}{(x-1)(x-1)}$$

$$f'(x)=\frac{-1}{(x-1)^2}$$

4) Find the derivate of $f(x)=\sqrt{x}$

Hint: If the function contains a radical, it may be necessary to use the conjugate to simplify the derivative.

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\bullet\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

$$f'(x)=\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$f'(x)=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$f'(x)=\frac{1}{\sqrt{x}+\sqrt{x}}$$

$$f'(x)=\frac{1}{2\sqrt{x}}$$

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Use the limit definition to find the derivative of each function.

1. $f(x) = x^3 - 2x$

2. $f(x) = \dfrac{3}{5x}$

3. $f(x) = ax^2 + bx + c$

4. $f(x) = \sqrt{x^2 + 4}$

5. $f(x) = \dfrac{x-1}{x+1}$

6. $f(x) = \dfrac{1}{\sqrt{3x}}$

7. $f(x) = \dfrac{6}{x^2 + 1}$

8. $f(x) = \dfrac{3}{\sqrt{x-2}}$

9. $f(x) = \dfrac{2x-1}{x-4}$

10. $f(x) = \dfrac{1}{x}$

$$f(x) = x^3 - 2x$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - 2(x+h) - (x^3 - 2x)}{h} =$$

$$\lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h - x^3 + 2x}{h} =$$

$$\lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 2h}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 2)}{h} =$$

$$\lim_{h \to 0} (3x^2 + 3xh + h^2 - 2) = 3x^2 - 2$$

$$f'(x) = 3x^2 - 2$$

$$f(x) = \frac{3}{5x}$$

$$\lim_{h \to 0} \frac{\frac{3}{5(x+h)} - \frac{3}{5x}}{h} = \lim_{h \to 0} \frac{\frac{3x}{5x(x+h)} - \frac{3(x+h)}{5x(x+h)}}{h} = \lim_{h \to 0} \frac{\frac{3x - 3(x+h)}{5x(x+h)}}{h} =$$

$$\lim_{h \to 0} \frac{\frac{3x - 3x - 3h}{5x(x+h)}}{h} = \lim_{h \to 0} \frac{\frac{-3h}{5x(x+h)}}{h} \cdot \frac{\frac{1}{h}}{\frac{1}{h}} = \lim_{h \to 0} \frac{-3}{5x(x+h)} =$$

$$\frac{-3}{5x(x)} = \frac{-3}{5x^2} \qquad f'(x) = \frac{-3}{5x^2}$$

$$f(x) = ax^2 + bx + c$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} =$$

$$\lim_{h\to 0} \frac{a(x^2 + 2xh + h^2) + bx + bh + c - ax^2 - bx - c}{h} =$$

$$\lim_{h\to 0} \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 - bx - c}{h} =$$

$$\lim_{h\to 0} \frac{2axh + ah^2 + bh}{h} = \lim_{h\to 0} \frac{h(2ax + ah + b)}{h} =$$

$$\lim_{h\to 0} (2ax + ah + b) = 2ax + b$$

$$f'(x) = 2ax + b$$

$$g(x) = \sqrt{x^2+4}$$

$$\lim_{h\to 0} \frac{\sqrt{(x+h)^2+4} - \sqrt{x^2+4}}{h} \cdot \frac{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}}{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}} =$$

$$\lim_{h\to 0} \frac{(x+h)^2+4 - (x^2+4)}{h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 4 - x^2 - 4}{h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)} =$$

$$\lim_{h\to 0} \frac{2xh + h^2}{h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)} = \lim_{h\to 0} \frac{h(2x+h)}{h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)} =$$

$$\lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}} = \frac{2x}{\sqrt{x^2+4} + \sqrt{x^2+4}} = \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}}$$

$$g'(x) = \frac{x}{\sqrt{x^2+4}} \quad \text{or} \quad \frac{x\sqrt{x^2+4}}{x^2+4}$$

$g(x) = \frac{x-1}{x+1}$

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h-1}{x+h+1} - \frac{x-1}{x+1}}{h} = \lim_{h\to 0} \frac{\frac{x+1}{x+1}\cdot\frac{x+h-1}{x+h+1} - \frac{x+h+1}{x+h+1}\cdot\frac{x-1}{x+1}}{h}$$

$$\lim_{h\to 0} \frac{\frac{(x+1)(x+h-1)-(x+h+1)(x-1)}{(x+1)(x+h+1)}}{h} = \lim_{h\to 0} \frac{\frac{x^2+xh-x+x+h-1-(x^2-x+xh-h+x-1)}{(x+1)(x+h+1)}}{h}$$

$$\lim_{h\to 0} \frac{\frac{x^2+xh-x+x+h-1-x^2+x-xh+h-x+1}{(x+1)(x+h+1)}}{h} = \lim_{h\to 0} \frac{\frac{2h}{(x+1)(x+h+1)}}{h} \cdot \frac{\frac{1}{h}}{\frac{1}{h}}$$

$$\lim_{h\to 0} \frac{2}{(x+1)(x+h+1)} = \frac{2}{(x+1)(x+1)} = \frac{2}{(x+1)^2}$$

$$g'(x) = \frac{2}{(x+1)^2}$$

$g(x) = \frac{1}{\sqrt{3x}}$

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{3(x+h)}} - \frac{1}{\sqrt{3x}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{3x}}{\sqrt{3x}}\cdot\frac{1}{\sqrt{3(x+h)}} - \frac{\sqrt{3(x+h)}}{\sqrt{3(x+h)}}\cdot\frac{1}{\sqrt{3x}}}{h}$$

$$\lim_{h\to 0} \frac{\frac{\sqrt{3x}-\sqrt{3(x+h)}}{\sqrt{3x}\sqrt{3(x+h)}}}{h} \cdot \frac{\sqrt{3x}+\sqrt{3(x+h)}}{\sqrt{3x}+\sqrt{3(x+h)}}$$

$$\lim_{h\to 0} \frac{\frac{3x-3(x+h)}{\sqrt{3x}\sqrt{3(x+h)}\left(\sqrt{3x}+\sqrt{3(x+h)}\right)}}{h} \quad \lim_{h\to 0} \frac{\frac{-3h}{\sqrt{3x}\sqrt{3(x+h)}\left(\sqrt{3x}+\sqrt{3(x+h)}\right)}}{h} \cdot \frac{\frac{1}{h}}{\frac{1}{h}}$$

$$\lim_{h\to 0} \frac{-3}{\sqrt{3x}\sqrt{3(x+h)}\left(\sqrt{3x}+\sqrt{3(x+h)}\right)} = \frac{-3}{\sqrt{3x}\sqrt{3x}\left(\sqrt{3x}+\sqrt{3x}\right)}$$

$$= \frac{-3}{3x\cdot 2\sqrt{3x}} = \frac{-1}{2x\sqrt{3x}}$$

$$g'(x) = \frac{-1}{2x\sqrt{3x}}$$

$$g(x) = \frac{6}{x^2+1}$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{6}{(x+h)^2+1} - \frac{6}{x^2+1}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{6}{(x^2+2xh+h^2+1)} \cdot \frac{(x^2+1)}{(x^2+1)} - \frac{6}{(x^2+1)} \cdot \frac{(x^2+2xh+h^2+1)}{(x^2+2xh+h^2+1)}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{6(x^2+1) - 6(x^2+2xh+h^2+1)}{(x^2+1)(x^2+2xh+h^2+1)}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{6x^2+6-6x^2-12xh-6h^2-6}{(x^2+1)(x^2+2xh+h^2+1)}}{h} = \lim_{h\to 0} \frac{\frac{-12xh-6h^2}{(x^2+1)(x^2+2xh+h^2+1)}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{h(-12x-6h)}{(x^2+1)(x^2+2xh+h^2+1)}}{\frac{h}{1}} \cdot \frac{\frac{1}{h}}{\frac{1}{h}} = \lim_{h\to 0} \frac{-12x-6h}{(x^2+1)(x^2+2xh+h^2+1)}$$

$$= \frac{-12x}{(x^2+1)(x^2+1)}$$

$$= \frac{-12x}{(x^2+1)^2}$$

$$g'(x) = \frac{-12x}{(x^2+1)^2}$$

$$g(x) = \frac{3}{\sqrt{x-2}}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{3}{\sqrt{x+h-2}} - \frac{3}{\sqrt{x-2}}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{\sqrt{x-2}}{\sqrt{x-2}} \cdot \frac{3}{\sqrt{x+h-2}} - \frac{\sqrt{x+h-2}}{\sqrt{x+h-2}} \cdot \frac{3}{\sqrt{x-2}}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{3\sqrt{x-2} - 3\sqrt{x+h-2}}{\sqrt{x-2}\ \sqrt{x+h-2}}}{h} \cdot \frac{3\sqrt{x-2}+3\sqrt{x+h-2}}{3\sqrt{x-2}+3\sqrt{x+h-2}} =$$

$$\lim_{h\to 0} \frac{\frac{9(x-2)-9(x+h-2)}{(\sqrt{x-2})(\sqrt{x+h-2})(3\sqrt{x-2}+3\sqrt{x+h-2})}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{9x-18-9x-9h+18}{(\sqrt{x-2})(\sqrt{x+h-2})(3\sqrt{x-2}+3\sqrt{x+h-2})}}{h} = \lim_{h\to 0} \frac{\frac{-9h}{(\sqrt{x-2})(\sqrt{x+h-2})(3\sqrt{x-2}+3\sqrt{x+h-2})}}{h} \cdot \frac{\frac{1}{h}}{\frac{1}{h}}$$

$$\lim_{h\to 0} \frac{-9}{(\sqrt{x-2})(\sqrt{x+h-2})(3\sqrt{x-2}+3\sqrt{x+h-2})} = \frac{-9}{(\sqrt{x-2})(\sqrt{x-2})(3\sqrt{x-2}+3\sqrt{x-2})} =$$

$$= \frac{-9}{(x-2)(6\sqrt{x-2})}$$

$$= \frac{-9}{6(x-2)\sqrt{x-2}}$$

$$= \frac{-3}{2(x-2)\sqrt{x-2}}$$

$$= \frac{-3}{2(x-2)(x-2)^{1/2}}$$

$$= \frac{-3}{2(x-2)^{3/2}}$$

$$g'(x) = \frac{-3}{2(x-2)^{3/2}}$$

5) $g(x) = \dfrac{2x-1}{x-4}$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{2(x+h)-1}{x+h-4} - \dfrac{2x-1}{x-4}}{h} =$$

$$\lim_{h \to 0} \frac{\dfrac{(2x+2h-1)}{(x+h-4)} \cdot \dfrac{(x-4)}{(x-4)} - \dfrac{(2x-1)}{(x-4)} \cdot \dfrac{(x+h-4)}{(x+h-4)}}{h} =$$

$$\lim_{h \to 0} \frac{\dfrac{(2x+2h-1)(x-4) - (2x-1)(x+h-4)}{(x+h-4)(x-4)}}{h}$$

$$\lim_{h \to 0} \frac{\dfrac{(2x^2+2xh-8x-8h-x+4) - (2x^2+2xh-8x-x-h+4)}{(x+h-4)(x-4)}}{h}$$

$$\lim_{h \to 0} \frac{\dfrac{2x^2+2xh-8x-8h-x+4-2x^2-2xh+8x+x+h-4}{(x+h-4)(x-4)}}{h}$$

$$\lim_{h\to 0} \frac{\frac{-7h}{(x+h-4)(x-4)}}{\frac{h}{1}} \cdot \frac{\frac{1}{h}}{\frac{1}{h}} = \lim_{h\to 0} \frac{-7}{(x+h-4)(x-4)}$$

$$= \frac{-7}{(x-4)(x-4)}$$

$$= \frac{-7}{(x-4)^2}$$

$$g'(x) = \frac{-7}{(x-4)^2}$$

$f(x) = \frac{1}{x}$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x}{x}\cdot\frac{1}{x+h} - \frac{x+h}{x+h}\cdot\frac{1}{x}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}} \cdot \frac{\frac{1}{h}}{\frac{1}{h}} =$$

$$\lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

$$f'(x) = \frac{-1}{x^2}$$